Recursive Sequences

MATHS Exam Paper 3

LIBERTY LIFE

Assessment Standard: 12.1.3
Correctly interpret recursive formulae

In this module we will introduce you to an alternative method of writing down the general term T_n of a number pattern or a sequence of numbers.

This method, known as using a recursive formula, is used in Computer Science and Advance Programme Mathematics in the Finance Module at school level.

A recursive formula is different to our other explicit formula, in that it expresses a given term T_n as a function which uses one or more of the previous terms. This means that you need to know at least the term value immediately before the term you are trying to find, (which can be quite restrictive).

Any recursive formula must have 2 parts:

- A starting value for at least T_1 (and sometimes T_2); and
- An equation for T_n as a function of T_{n-1} (or T_{n+1} as a function of T_n).

Collect your Paper 3 Lessons every week!!

Guys, both NSC and IEB examinations candidates have the option of writing Paper 3 at the end of the year! Paper 3 covers additional mathematics material and is out of 100 marks. Maths Paper 3 will really set you apart in the job market, and make studying technical subjects at tertiary level easier. We have hooked you up with these lessons - written by IEB Maths Paper 3 examiner Heather Frankiskos. Though the lessons apply to both IEB and NSC candidates, where there are differences, we will point them out! The lesson this week applies to candidates from both examining bodies. **Give it a go!**

Let us look at a recursive formula you should be familiar with: the one that generates the well-known **Fibonacci Sequence** where each term in the sequence from T_3 onwards is the sum of the previous two terms.

1 ; 1 ; 2 ; 3 ; 5 ; 8 ; 13; ...

We would write this as

$T_{n+1} = T_n + T_{n-1}$ (But we have to state that $T_1 = 1$ and $T_2 = 1$.)

If we state that $T_1 = 1$ and $T_2 = 3$ then the sequence generated by $T_{n+1} = T_n + T_{n-1}$ would be:

1 ; 3 ; 4 ; 7 ; 11 ; 18 ; ... (Which is completely different)

Activity 1

If you are given a recursive formula you must be able to generate the sequence:

Example 1

Given $T_1 = 6$ and $T_{n+1} = T_n + 5$ determine the first four terms of the sequence.

Solution:

$T_2 = T_1 + 5 \quad = \quad 6 + 5 \quad = \quad 11$

$T_3 = T_2 + 5 \quad = \quad 11 + 5 \quad = \quad 16$

$T_4 = T_3 + 5 \quad = \quad 16 + 5 \quad = \quad 21$

Therefore the sequence is: 6 ; 11 ; 16 ; 21; ...

Note:

- We could have written this in explicit form as, $T_n = 5n + 6$.
- This is a first order recursive formula using only one immediate term before it, and a constant difference is added.
- If we needed T_{10} we would first need to work out T_9 ; T_8 ; T_7 ; T_6 ; T_5 before it. Quite a mission!

Example 2

Given $T_{n+1} = T_n + 2n$ with $T_1 = 3$ determine the first four terms of the sequence.

Solution

$T_2 = T_1 + 2\,(1) \quad = \quad 3 + 2 \quad = \quad 5$

$T_3 = T_2 + 2\,(2) \quad = \quad 5 + 4 \quad = \quad 9$

$T_4 = T_3 + 2\,(3) \quad = \quad 9 + 6 \quad = \quad 15$

Therefore the sequence is: 3 ; 5 ; 9 ; 15 ; ...

Which we know would be written in the explicit form as $T_n = n^2 - n + 3$

Example 3

Find the next four terms of a sequence given

$T_{n+1} \quad = \quad T_n + T_{n-1} + 3$ with $T_1 = 1$; $T_2 = 2$

Note:

We must give the value of two previous terms to use this recursive formula.

Solution:

$T_3 = T_2 + T_{1-} + 3 = 2 + 1 + 3 = 6$
$T_4 = T_3 + T_2 + 3 = 6 + 2 + 3 = 11$
$T_5 = T_4 + T_3 + 3 = 11 + 6 + 3 = 20$
$T_6 = T_5 + T_4 + 3 = 20 + 11 + 3 = 34$

Sequence is 1 ; 2 ; 6 ; 11 ; 20 ; 34 ; ...

Imagine trying to find T_{30} - not such fun! This is the drawback of the recursive formula

Activity 2

If you are given a recursive sequence you must be able to derive the formula:

Example 1

Consider the sequence: 2 ; -6 ; -14 ; -22 ; ... and give a recursive formula in the form $T_n = ...$

Solution:

$T_1 = 2$
$T_n = T_{n-1} - 8$ for $n \geq 2$ or $T_{n+1} = T_n - 8$ for $n \geq 1$

Example 2

Consider the sequence: 2 ; 5 ; 6 ; 10 ; 15 ; 24 ; 38 ; 61 ; ...
and write down a recursive rule to describe T_{n+1} in terms of one or more of its preceding terms

Solution

$T_{n+1} = T_n + T_{n-1} - 1$ for $T_1 = 2$ and $T_2 = 5$

Note:

From T_3 onwards, each term is in the sum of the previous two terms, reduced by 1.

Example 3

1 ; 2 ; 2 ; 4 ; 8 ; 32 ; 256 ; . . . is a recursive sequence.
Write down the general rule in the form of $T_{n+1} = ...$

Solution

Notice that from T_3 onwards, each term is the product of the previous two terms.

$$T_{n+1} = T_n \times T_{n-1} \; ; \quad T_1 = 1 \text{ and } T_2 = 2$$

So you are now ready for an examination type question. This one combines explicit and recursive formula so it should really test your understanding of this section.

Activity 3

Given the recursive rule $T_{n+1} = T_n + 3n$; $T_1 = 1$

a) Find the next four terms of the sequence
b) Hence give the explicit formula for this sequence in the form $T_n = an^2 + bn + c$

Solution

a)
$T_2 = 1 + 3(1) = 4$
$T_3 = 4 + 3(2) = 10$
$T_4 = 10 + 3(3) = 19$
$T_5 = 19 + 3(4) = 31$

So sequence is 1 ; 4 ; 10 ; 19 ; 31 ; ...

b) 1 ; 4 ; 10 ; 19 ; 31 ; ....

3 6 9 12

3 3 3

Now $2a = 3$

$a = \frac{3}{2}$

now $T_1 = \frac{3}{2}(1)^2 + b(1) + c = 1$ and $T_2 = \frac{3}{2}(2)^2 + b(2) + c = 4$

Solving for b and c we get $b = -\frac{3}{2}$ and $c = 1$

So $T_n = \frac{3}{2}n^2 - \frac{3}{2}n + 1$ is the explicit formula